Math 150, *Lecture Notes- Bonds* Name____________

Section 5.5 *Bases Other than e and Applications*

Bases Other than *e*

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

Definition of Exponential Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

These functions obey the usual laws of exponents. For instance, here are some familiar properties.

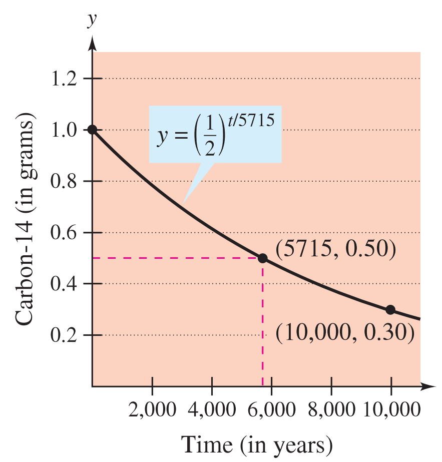
1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$

3. $\dfrac{a^x}{a^y} = a^{x-y}$ **4.** $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (*Half-life* is the number of years required for half of the atoms in a sample of radioactive material to decay.)

Ex.1 Radioactive Half-Life Model

The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?



The half-life of carbon-14 is about 5715 years.
Figure 5.25

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

Definition of Logarithmic Function to Base a

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

NOTE In precalculus, you learned that $\log_a x$ is the value to which a must be raised to produce x. This agrees with the definition given here because

$$\begin{aligned} a^{\log_a x} &= a^{(1/\ln a)\ln x} \\ &= (e^{\ln a})^{(1/\ln a)\ln x} \\ &= e^{(\ln a/\ln a)\ln x} \\ &= e^{\ln x} \\ &= x. \end{aligned}$$

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function given in Theorem 5.2. (Assume x and y are positive numbers and n is rational.)

1. $\log_a 1 = 0$ — Log of 1
2. $\log_a xy = \log_a x + \log_a y$ — Log of a product
3. $\log_a x^n = n \log_a x$ — Log of a power
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$ — Log of a quotient

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x, \quad \text{for } x > 0$
3. $\log_a a^x = x, \quad \text{for all } x$

The logarithmic function to the base 10 is called the **common logarithmic function.** So, for common logarithms, $y = 10^x$ if and only if $x = \log_{10} y$.

Ex.2 Bases other than e

Solve for x in each equation.

a. $3^x = \dfrac{1}{81}$ **b.** $\log_2 x = -4$

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the following differentiation rules for bases other than e.

THEOREM 5.13 Derivatives for Bases Other Than e

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$

2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u \dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$

4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u} \dfrac{du}{dx}$

PROOF By definition, $a^x = e^{(\ln a)x}$. So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u \frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules. ■

Ex.3 Differentiating Functions to Other Bases

Find the derivative of each function.

a. $y = 2^x$

b. $y = 2^{3x}$

c. $y = \log_{10} \cos x$

Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options: (1) convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration formula

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

(which follows from Theorem 5.13).

Ex.4 Integrating an Exponential Function to Another Base

Find $\int 2^x\,dx$.

When the Power Rule, $D_x[x^n] = nx^{n-1}$, was introduced in Chapter 2, the exponent n was required to be a rational number. Now the rule is extended to cover any real value of n. Try to prove this theorem using logarithmic differentiation.

THEOREM 5.14 The Power Rule for Real Exponents

Let n be any real number and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[x^n] = nx^{n-1}$
2. $\dfrac{d}{dx}[u^n] = nu^{n-1}\dfrac{du}{dx}$

The next example compares the derivatives of four types of functions. Each function uses a different differentiation formula, depending on whether the base and the exponent are constants or variables.

Ex.5 Comparing Variables and Constants

a. $\frac{d}{dx}[e^e] = 0$ Constant Rule

b. $\frac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\frac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Logarithmic differentiation

Applications of Exponential Functions

Suppose P dollars is deposited in an account at an annual interest rate r (in decimal form). If interest accumulates in the account, what is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

n	A
1	\$1080.00
2	\$1081.60
4	\$1082.43
12	\$1083.00
365	\$1083.28

For instance, the result for a deposit of \$1000 at 8% interest compounded n times a year is shown in the upper table at the left.

As n increases, the balance A approaches a limit. To develop this limit, use the following theorem. To test the reasonableness of this theorem, try evaluating $[(x+1)/x]^x$ for several values of x, as shown in the lower table at the left. (A proof of this theorem is given in Appendix A.)

THEOREM 5.15 A Limit Involving e

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

Now, let's take another look at the formula for the balance A in an account in which the interest is compounded n times per year. By taking the limit as n approaches infinity, you obtain

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^n \qquad \text{Take limit as } n\to\infty.$$

$$= P\lim_{n\to\infty}\left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^r \qquad \text{Rewrite.}$$

$$= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^r \qquad \text{Let } x = n/r. \text{ Then } x\to\infty \text{ as } n\to\infty.$$

$$= Pe^r. \qquad \text{Apply Theorem 5.15.}$$

This limit produces the balance after 1 year of **continuous compounding.** So, for a deposit of \$1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$A = 1000e^{0.08}$$
$$\approx \$1083.29.$$

These results are summarized below.

SUMMARY OF COMPOUND INTEREST FORMULAS

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (decimal form), and n = number of compoundings per year.

1. Compounded n times per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

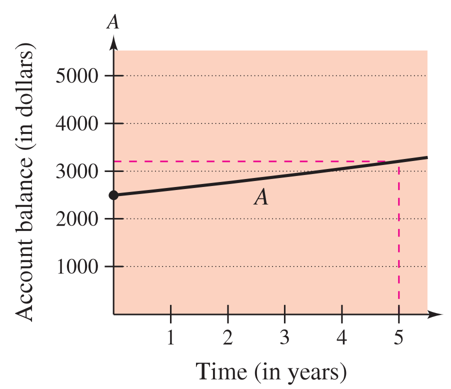
Ex.6 Comparing Continuous, Quarterly, and Monthly Compounding

A deposit of $2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the account at the end of 5 years if the interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

a. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.05}{4}\right)^{4(5)}$ Compounded quarterly

b. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.05}{12}\right)^{12(5)}$ Compounded monthly

c. $A = Pe^{rt} = 2500[e^{0.05(5)}]$ Compounded continuously



The balance in a savings account grows exponentially.
Figure 5.26